

CONFORMAL GEODESICS*

BY

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1. **Introduction.** The totality of extremals in a Riemann space V_n † connected with a calculus of variations problem of the form

$$(1.1) \quad \delta \int F ds = 0,$$

where F is a point function and ds is the element of length of V_n , constitutes an important family of $\infty^{2(n-1)}$ curves. Consider, for example, a conservative dynamical system for which neither the constraints nor the work function W involve the time. By the principle of least action the dynamical trajectories of a particle are the extremals of (1.1) with $F = [2m(c+W)]^{1/2}$ where m and c are the mass of the particle and the energy constant, respectively. Again, if ν is the index of refraction of an isotropic nonhomogeneous medium, the paths of light through this medium are the solutions of (1.1) with $F = \nu$ in accordance with Fermat's principle. Finally, let V_n and \bar{V}_n be two conformal Riemann spaces so that $d\bar{s} = e^\sigma ds$. Then the images of the geodesics of \bar{V}_n in V_n are the extremals of (1.1) where $F = e^\sigma$.‡

As we are interested primarily in the last interpretation, following Schouten,§ we call any family of $\infty^{2(n-1)}$ curves which is a solution of (1.1) a family of *conformal geodesics*. Of course, by a change of language, the theorems obtained have equal validity for the dynamical, optical, and other interpretations. The following topics are discussed and the corresponding questions answered in this paper:

I. A complete geometric characterization of the conformal geodesics of any Riemann space.

II. Additional special properties characteristic of conformal geodesics which are the images of the geodesics of a particular Riemann space (flat space, space of constant curvature, Einstein space).

* Presented to the Society, February 26, 1938; received by the editors June 15, 1938.

† We denote an n -dimensional Riemann space, Einstein space, and space of constant curvature by V_n , E_n , and S_n , respectively.

‡ These and other interpretations are discussed by E. Kasner, *Natural families of trajectories: conservative fields of force*, these Transactions, vol. 10 (1909), pp. 201–203. Also cf. L. P. Eisenhart, *Continuous Groups of Transformations*, 1933, pp. 277–280.

§ J. A. Schouten, *Über die Umkehrung eines Satzes von Lipschitz*, Nieuw Archief voor Wiskunde, vol. 15 (1928), pp. 97–102.

III. Some relations between the conformal geodesics of a Riemann space and of its subspaces.

IV. Some special geometric problems.

Other questions concerning conformal geodesics and other extremals (the theorems of Lipschitz, Thomson and Tait, Kneser, and their converses) have been investigated by Kasner, Lipke, Schouten, Blaschke, Douglas, LaPaz, and Radó.* The first of the above topics was previously considered from the standpoint of dynamics and a solution obtained by Kasner† for the case of a euclidean space and by Lipke‡ for a Riemann space whose first fundamental form is positive definite. The characterization which they obtained is stated under more general conditions in Theorem 2.1 (or 2.2) and Theorem 3.2. The method which is used in the present paper differs from that hitherto employed.

The geometry of conformal geodesics is closely related to the more general investigation of the geometric properties of any curves or subspaces of V_n and \bar{V}_n , respectively, which correspond under the given conformal transformation. Somewhat similar studies of some phases of this problem have recently been made by a number of writers.§

I. GEOMETRIC CHARACTERIZATION

2. **Property one: the principal normal.** Let V_n and \bar{V}_n be two conformal n -dimensional Riemann spaces whose first fundamental forms are||

$$(2.1) \quad ds^2 = g_{ij}dx^i dx^j,$$

$$(2.2) \quad d\bar{s}^2 = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j,$$

respectively, so that

$$(2.3) \quad d\bar{s} = e^\sigma ds.$$

* For references to all of these writers cf. L. LaPaz and T. Radó, *On a converse of Kneser's transversality theorem*, *Annals of Mathematics*, (2), vol. 36 (1935), pp. 749-769.

† E. Kasner, loc. cit., pp. 201-219.

‡ J. Lipke, *Natural families of curves in a general curved space of n dimensions*, these *Transactions*, vol. 13 (1912), pp. 77-95.

§ S. Sasaki, *Some theorems on conformal transformations of Riemannian spaces*, *Proceedings of the Physico-Mathematical Society of Japan*, IIIs, vol. 18 (1936), pp. 572-578, and V. Modesitt, *Some singular properties of conformal transformations between Riemann spaces*, *American Journal of Mathematics*, vol. 60 (1938), pp. 325-336. Also see an abstract by the author entitled *Conformal transformations and the subspaces of a Riemann space*, *Bulletin of the American Mathematical Society*, abstract 43-9-328.

|| Throughout this paper except where otherwise stated Latin indices have the range 1, 2, \dots , n . An index which appears twice in an expression is to be summed over the appropriate range unless the index appears in parentheses. A free index in a tensor equation assumes each value of its range.

It is assumed that these forms are not singular although they may be indefinite. We choose coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$ so that the conformal correspondence becomes $\bar{x}^i = x^i$. In these coordinate systems

$$(2.4) \quad \bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij},$$

where g^{ij} and \bar{g}^{ij} are the contravariant components of the metric tensors. If the Christoffel symbols of the second kind for V_n and \bar{V}_n are written $\{k|ij\}$ and $\{\bar{k}|\bar{i}\bar{j}\}$, respectively, it follows from (2.4) that*

$$(2.5) \quad \{\bar{k}|\bar{i}\bar{j}\} = \{k|ij\} + \delta_i^k \sigma_{,j} + \delta_j^k \sigma_{,i} - g_{ij} g^{km} \sigma_{,m}.$$

Let C be a curve in V_n and \bar{C} its image in \bar{V}_n . If the unit tangents to C and \bar{C} at corresponding points are denoted by ξ^i and $\bar{\xi}^i$ and the principal normals by μ^i and $\bar{\mu}^i$, it is an easy consequence of (2.4) and (2.5) that

$$(2.6) \quad \bar{\xi}^i = e^{-\sigma} \xi^i,$$

$$(2.7) \quad \bar{\mu}^i = e^{-2\sigma} [\mu^i - e \sigma_{,m} (g^{im} - e \xi^i \xi^m)],$$

where e is $+1$ or -1 , being determined by $e = g_{ij} \xi^i \xi^j$.

Since $\bar{\mu}^i = 0$ for the geodesics of \bar{V}_n , the characteristic equation of a family of conformal geodesics in V_n is, according to (2.7),

$$(2.8) \quad \mu^i = e \sigma_{,m} (g^{im} - e \xi^i \xi^m).$$

In the derivation of this equation and throughout the paper we exclude those conformal geodesics which are tangent to null vectors. Now $g^{im} - e \xi^i \xi^m$ is the projection tensor† for the vector space normal to ξ^i . Hence we have as a result of (2.8) the following theorem:

THEOREM 2.1. *The principal normal of any curve of a family of conformal geodesics passing through a common point in a non-null direction is, except for sign, the projection normal to this direction of a fixed vector; the sign is determined by the character of the tangent to the curve.*

This is the first characteristic property of conformal geodesics. Of course the fixed vector is the gradient $\sigma_{,i}$. As a consequence of Theorem 2.1, we have the following equivalent theorem:

THEOREM 2.2. *If the curves of a family of conformal geodesics which pass through a common point of V_n are projected orthogonally upon the tangent flat S_n at that point, the centers of curvature of the ∞^{n-1} projections at the common point*

* The comma denotes covariant differentiation with respect to the x 's and the form (2.1), and the δ_i^k are the Kronecker deltas.

† Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, vol. 2, 1930, pp. 44-45.

will lie on a flat S_{n-1} orthogonal to the fixed vector of Theorem 2.1, and the ∞^{n-1} osculating circles of the projections will have a second point in common.

The proof is immediate. For the principal normals of the conformal geodesics in V_n are identical with the principal normals of their orthogonal projections in the flat S_n . From Theorem 2.1, it follows that the end points of the principal normals of the projections lie on a spherical S_{n-1} whose diameter is the length of the gradient $\sigma_{,i}$. By inversion, it is seen that the centers of curvature lie on a hyperplane of S_n normal to the direction of $\sigma_{,i}$. According to (2.8), the linear vector space determined by the tangent and principal normal of any curve of (2.8) contains $g^{im}\sigma_{,m}$. This proves the next theorem:

THEOREM 2.3. *The osculating geodesic surfaces of the curves of a family of conformal geodesics which pass through a common point form a bundle of surfaces; they all contain the fixed vector of Theorem 2.1.*

If we omit the condition that the fixed vector of Theorem 2.1 be a gradient, it follows easily that Theorem 2.1 is the characteristic property of all families of curves whose equations are of the form

$$(2.9) \quad \mu^i = e\tau_m(g^{im} - e\xi^i\xi^m),$$

where τ_i is an arbitrary vector. The solutions of (2.9) have been called *velocity systems* because of their connection with motion in fields of force. A geometric definition of velocity systems is possible. For consider the Weyl geometry* whose coefficients of connection Γ_{jk}^i are

$$\Gamma_{jk}^i = \{i|jk\} + \delta_j^i\tau_k + \delta_k^i\tau_j - g_{jk}g^{im}\tau_m.$$

Then if the points in this Weyl space and V_n which have the same coordinates correspond, it follows easily that the velocity system (2.9) consists of the images in V_n of the paths of this Weyl geometry. It is clear that Theorems 2.1, 2.2, and 2.3 hold for all velocity systems.

3. Property two: hyperosculating geodesic circles. Velocity systems are characterized by Theorem 2.1. It remains to distinguish geometrically the families of conformal geodesics among the totality of velocity systems. For this purpose, we shall consider the osculating geodesic circles of the curves.

Let C be a curve in V_n , and denote the unit tangent, and unit normals of orders 1, 2, \dots , $n-1$, and the first, second, \dots , $(n-1)$ st curvatures of C by ${}_{(1)}\xi^i$, and ${}_{(2)}\xi^i$, ${}_{(3)}\xi^i$, \dots , ${}_{(n)}\xi^i$, and k_1, k_2, \dots, k_{n-1} , respectively. A geodesic

* Such geometries were proposed by Weyl as the basis of a combined theory of gravitation and electro-dynamics. Cf. H. Weyl, *Space, Time, Matter*, English translation, 1921, pp. 125, 296, and L. P. Eisenhart, *Non-Riemannian Geometry*, American Mathematical Society Colloquium Publications, vol. 8, New York, 1927, pp. 81-82.

circle of V_n is defined as a curve whose first curvature in V_n is constant and whose second curvature is identically zero. The geodesic circle which is tangent to C and has the same first curvature as C at the point of tangency is called the *osculating geodesic circle* of C at the point.* It follows from a fundamental existence theorem of differential equations that every curve for which ${}_{(1)}\xi^i$, ${}_{(2)}\xi^i$, and k_1 exist at a point has a unique osculating geodesic circle at that point.†

The Frenet equations of C are

$$(3.1) \quad \frac{\delta {}_{(m)}\xi^i}{\delta s} = -e_{m-1}k_{m-1} {}_{(m-1)}\xi^i + e_{m+1}k_m {}_{(m+1)}\xi^i,$$

$$m = 1, 2, \dots, n-1; k_0 = 0,$$

where

$$(3.2) \quad e_m = g_{ij} {}_{(m)}\xi^i {}_{(m)}\xi^j$$

and

$$(3.3) \quad g_{ij} {}_{(k)}\xi^i {}_{(m)}\xi^j = 0, \quad m \neq k,$$

and where $\delta/\delta s$ denotes covariant differentiation with respect to arc length along C so that

$$\frac{\delta \lambda^i}{\delta s} = \frac{d\lambda^i}{ds} + \{i|jk\} \lambda^j {}_{(1)}\xi^k; \quad \frac{\delta \lambda_i}{\delta s} = \frac{d\lambda_i}{ds} - \{j|ik\} \lambda_j {}_{(1)}\xi^k.$$

The geodesic circle of C will have higher than second order contact; that is, it will hyperosculate C if and only if the values of $\delta x^i/\delta s$, $\delta^2 x^i/\delta s^2$, and $\delta^3 x^i/\delta s^3$ are the same at the point of tangency. Since $\delta x^i/\delta s = {}_{(1)}\xi^i$, the Frenet equations (3.1) show that these conditions are equivalent to‡

$$(3.4) \quad \frac{dk_1}{ds} = 0, \quad k_2 = 0$$

at the point of contact.

Since $\xi^i = {}_{(1)}\xi^i$, $e = e_1$, and $\mu^i = e_2 k_1 {}_{(2)}\xi^i$, it follows from (2.9), (3.2), and

* Lipke defines the osculating geodesic circle of C at a point as the curve of constant geodesic curvature which lies in the osculating geodesic V_2 of C at the point and agrees with C in curvature and direction at the point. The results of §3 are valid for either definition of the osculating circle, but in later sections our present definition, based on the Frenet equations of a curve in V_n rather than in V_2 , is more advantageous.

† Cf. Duschek-Mayer, loc. cit., pp. 62-64, for this theorem and the subsequent use of the Frenet equations. If k_p is identically zero on a curve, it is to be understood in (3.1) that $k_{p+1} = \dots = k_{n-1} = 0$ and that ${}_{(p+1)}\xi^i, \dots, {}_{(n)}\xi^i$ are any vectors which satisfy (3.2) and (3.3).

‡ If $n=2$, the second of these equations should be omitted. Similar deletions are to be understood in equations (3.5) and (3.6).

(3.3) that for any velocity system

$$(3.5) \quad \tau_{i(2)}\xi^i = e_1 k_1, \quad \tau_{i(r)}\xi^i = 0, \quad r > 2.$$

If we differentiate (3.5) covariantly with respect to s and use (3.1) and (3.5), we find

$$(3.6) \quad \begin{aligned} (\tau_{i,j} - \tau_i \tau_j)_{(2)}\xi^i_{(1)}\xi^j &= e_1 \frac{dk_1}{ds}, \\ (\tau_{i,j} - \tau_i \tau_j)_{(3)}\xi^i_{(1)}\xi^j &= e_1 e_2 k_1 k_2, \\ (\tau_{i,j} - \tau_i \tau_j)_{(1+r)}\xi^i_{(1)}\xi^j &= 0, \end{aligned} \quad r > 2,$$

since $d\tau_i/ds = \tau_{i,j}\xi^j_{(1)}$.

According to (3.4) and (3.6), the directions $_{(1)}\xi^i$ at a fixed point in which velocity curves are hyperosculated by their osculating geodesic circles are given by

$$(3.7) \quad \tau_{ij(s)}\xi^i_{(1)}\xi^j = 0, \quad s > 1,$$

where

$$(3.8) \quad \tau_{ij} = \tau_{i,j} - \tau_i \tau_j.$$

The tensor τ_{ij} is symmetric when and only when τ_i is a gradient $\sigma_{,i}$. We call the directions in which hyperosculation occurs the *H-directions* of the velocity system. If we write $e_1 \rho_1 = \tau_{ij(1)}\xi^i_{(1)}\xi^j$, it follows from (3.2), (3.3), and (3.7) that

$$(3.9) \quad (\tau_{ij} - \rho_1 g_{ij})_{(m)}\xi^i_{(1)}\xi^j = 0.$$

Since the vectors $_{(m)}\xi^i$ are independent,

$$(\tau_{ij} - \rho_1 g_{ij})_{(1)}\xi^j = 0,$$

so that ρ_1 is a root of the determinant equation $|\tau_{ij} - \rho g_{ij}| = 0$, and $_{(1)}\xi^i$ is a principal direction determined by τ_{ij} . In general there are n distinct principal directions. This proves the following theorem:

THEOREM 3.1. *The H-directions of any velocity system are identical with the principal directions determined by the tensor (3.8) which are not tangent to null vectors. The velocity system is a family of conformal geodesics if and only if this tensor is symmetric.*

If τ_{ij} is a symmetric tensor and none of the corresponding principal directions are null vectors (as is always the case if (2.1) is definite), it follows from the known theory* that there exist n mutually orthogonal non-null principal

* L. P. Eisenhart, *Riemannian Geometry*, 1926, pp. 107-112.

directions. Conversely, let ${}_{(p)}\lambda^i$ be n principal directions determined by τ_{ij} such that $g_{ij} {}_{(p)}\lambda^i {}_{(q)}\lambda^j = 0$ if $p \neq q$. Then $\tau_{ij} {}_{(p)}\lambda^i {}_{(q)}\lambda^j = 0$, ($p \neq q$). Any two arbitrary vectors α^i, β^i may be written as

$$\alpha^i = \sum_{t=1}^n a_t {}_{(t)}\lambda^i, \quad \beta^i = \sum_{t=1}^n b_t {}_{(t)}\lambda^i,$$

where a_t and b_t are constants so that

$$\tau_{ij} \alpha^i \beta^j = \sum_{t=1}^n a_t b_t \tau_{ij} {}_{(t)}\lambda^i {}_{(t)}\lambda^j = \tau_{ij} \beta^i \alpha^j.$$

Hence τ_{ij} is a symmetric tensor so that $\tau_i = \sigma_{,i}$. This completes the characterization of conformal geodesics stated in the next theorem:

THEOREM 3.2. *A family of conformal geodesics in a V_n whose first fundamental form is definite admits an orthogonal ennuple of H -directions at each point. Conversely, if a velocity system in any V_n admits n mutually orthogonal H -directions at each point, it is a family of conformal geodesics.*

4. The H -directions. We consider the H -directions of a family of conformal geodesics in greater detail. If the conformal correspondence between V_n and \bar{V}_n is given by (2.3), in accordance with Theorem 3.1, the H -directions of the images in V_n of the geodesics of \bar{V}_n coincide with the non-null principal directions determined by the tensor

$$(4.1) \quad \sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}.$$

The H -directions of the images in \bar{V}_n of the geodesics of V_n are similarly determined by the tensor*

$$(4.2) \quad \bar{\sigma}_{ij} = (-\sigma)_{,ij} - (-\sigma)_{,i}(-\sigma)_{,j}.$$

It follows readily from (2.5), (4.1), and (4.2) that

$$(4.3) \quad \bar{\sigma}_{ij} = -\sigma_{ij} - \Delta_1 \sigma g_{ij},$$

where $\Delta_1 \sigma = g^{ij} \sigma_{,i}\sigma_{,j}$.

Now (4.3) is an equation of the form†

$$(4.4) \quad \nu_{ij} = \sum_{k=1}^m a_k {}_{(k)}\nu_{ij} + b g_{ij},$$

where the ${}_{(k)}\nu_{ij}$ and ν_{ij} are symmetric tensors of the second order and the a_k

* The semicolon here denotes covariant differentiation with respect to the x 's and the form (2.2).

† Here (k) denotes the tensor and ij the components.

and b are scalars. If λ^i is a common principal direction in V_n determined by each of the ${}_{(k)}\nu_{ij}$, quantities ρ_k exist such that

$$(4.5) \quad ({}_{(k)}\nu_{ij} - \rho_k g_{ij})\lambda^i = 0, \quad k = 1, 2, \dots, m.$$

It follows from (4.4) and (4.5) that $(\nu_{ij} - \rho g_{ij})\lambda^i = 0$, where $\rho = \sum_{k=1}^m a_k \rho_k + b$, so that λ^i is also a principal direction determined by ν_{ij} .

This result shows that the principal directions in V_n determined by σ_{ij} and $\bar{\sigma}_{ij}$ coincide. Furthermore, according to (2.4), the principal directions determined by any tensor in V_n and the tensor having the same components in \bar{V}_n correspond. Hence the H -directions in V_n and \bar{V}_n correspond by means of the conformal transformation. Since the H -directions in V_n and \bar{V}_n are conformally equivalent, the mapping determines a unique set of H -directions. We call these H -directions the *H-directions of the conformal transformation* (2.3).

If R_{hijk} and \bar{R}_{hijk} are the Riemann curvature tensors of V_n and \bar{V}_n , it follows from (2.5) by straightforward calculation* that

$$(4.6) \quad e^{-2\sigma} \bar{R}_{hijk} = R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj} \\ + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma.$$

By means of (2.4) and (4.6), we find

$$(4.7) \quad (n-2)\sigma_{ij} = \bar{R}_{ij} - R_{ij} - g_{ij}[\Delta_2\sigma + (n-2)\Delta_1\sigma],$$

where R_{ij} and \bar{R}_{ij} are the Ricci tensors of V_n and \bar{V}_n , respectively, and $\Delta_2\sigma = g^{ij}\sigma_{ij}$. Thus, if $n > 2$, the H -directions of the conformal transformation are the non-null principal directions determined by $\bar{R}_{ij} - R_{ij}$. We state these results in the following theorem:

THEOREM 4.1. *Let V_n and \bar{V}_n be two conformal Riemann spaces. Then the H -directions of the images in V_n of the geodesics of \bar{V}_n and of the images in \bar{V}_n of the geodesics of V_n correspond under the mapping. If $n > 2$, these H -directions coincide with the principal directions determined by $R_{ij} - \bar{R}_{ij}$ which are not tangent to null vectors.*

Since (4.7) is of the form (4.4), we conclude that if $n > 2$ a principal direction determined by two of the tensors σ_{ij} , R_{ij} , \bar{R}_{ij} is also determined by the third. This proves the next theorem:

THEOREM 4.2. *Let V_n and \bar{V}_n be two conformal Riemann spaces of dimensionality $n > 2$. Then by the mapping of V_n on \bar{V}_n , a non-null Ricci principal direction of V_n corresponds to a Ricci principal direction of \bar{V}_n if and only if it is an H -direction of the conformal transformation.*

* For example, cf. Eisenhart, loc. cit., pp. 89-90.

II. PARTICULAR RIEMANN SPACES

5. **Conjugate conformal geodesics at a point.** Let λ^i be any unit vector at a point P of V_n which is not tangent to $\sigma_{,i}$. Then there exists a unique unit vector* λ'^i in the linear vector space at P determined by λ^i and $\sigma_{,i}$ such that

$$g_{ij}\lambda^i\lambda'^j = 0.$$

Let C and C' be the curves belonging to a family of conformal geodesics which are tangent to λ^i and λ'^i , respectively. Then C' is called the *conjugate conformal geodesic* of C at P . If $\lambda^i\sigma_{,i} \neq 0$, it is clear that this relationship is reciprocal, so that we may speak of C and C' as conjugate conformal geodesics. If $\lambda^i\sigma_{,i} = 0$, the conjugate conformal geodesic of C' is not defined.

According to Theorem 2.1, if $\lambda^i\sigma_{,i} \neq 0$, the unit first normals of C and C' at P are λ'^i and λ^i , respectively. Hence conjugate conformal geodesics have the same osculating geodesic surface. From (3.5), we have at P

$$(5.1) \quad \sigma_{,i}\lambda'^i = ek, \quad \sigma_{,i}\lambda^i = e'k',$$

where k and k' are the first curvatures of C and C' , respectively, and

$$(5.2) \quad e = g_{ij}\lambda^i\lambda^j, \quad e' = g_{ij}\lambda'^i\lambda'^j.$$

If (2.8) is multiplied by $\sigma_{,i}$ and summed for i , it follows from (5.1) that

$$(5.3) \quad ek'^2 + e'k^2 = \Delta_1\sigma.$$

If C is any conformal geodesic orthogonal to $\sigma_{,i}$ at P , the equation corresponding to (5.3) is

$$(5.4) \quad e_\sigma k_\sigma^2 = \Delta_1\sigma,$$

where k_σ is the first curvature of C and where e_σ is +1 or -1 according as $\Delta_1\sigma$ is positive or negative. If $\Delta_1\sigma = 0$, $k_\sigma = 0$.

As P moves along C , we obtain by covariant differentiation of (5.1) with respect to s , after using (3.1), (3.5), (5.3), and (5.4), that

$$(5.5) \quad \sigma_{ij}\lambda'^i\lambda^j = e \frac{dk}{ds},$$

$$(5.6) \quad \sigma_{ij}\lambda^i\lambda^j = e' \frac{dk'}{ds} - ee_\sigma k_\sigma^2,$$

where σ_{ij} is defined by (4.1). Since σ_{ij} is a symmetric tensor, if $\lambda^i\sigma_{,i} \neq 0$ it follows from (5.5) that $edk/ds = e'dk'/ds'$, where s' denotes the arc length

* An exception occurs only if $\lambda^i\sigma_{,i} = 0$ and $\Delta_1\sigma = 0$. In this case, λ'^i is tangent to $\sigma_{,i}$ and is a null vector.

of C' . If $n=2$, it can be shown that this last equation is equivalent to the second characteristic property of conformal geodesics.

6. Conformal images of the geodesics of an S_n or E_n . The geometric characterization of any family of conformal geodesics in V_n is derived in §§2 and 3. If these curves are the images of the geodesics of a space of constant curvature or an Einstein space, they will enjoy additional special properties which are obtained below. Let ${}_{(\alpha)}\lambda^i$ and ${}_{(\beta)}\lambda^i$ be any two unit vectors in V_n at a point P neither of which is tangent to $\sigma_{,i}$. We denote by C_α and C_β the conformal geodesics tangent to these vectors and by C'_α and C'_β the corresponding conjugate conformal geodesics.

According to (2.6) and (4.6),

$$(6.1) \quad e^{2\sigma}\bar{r}_{\alpha\beta} = r_{\alpha\beta} - e_\beta\sigma_{ik}{}_{(\beta)}\lambda^i{}_{(\beta)}\lambda^j - e_\alpha\sigma_{hj}{}_{(\alpha)}\lambda^h{}_{(\alpha)}\lambda^j - \Delta_1\sigma,$$

where $r_{\alpha\beta}$ and $\bar{r}_{\alpha\beta}$ are the Riemannian curvatures at P of V_n and \bar{V}_n , respectively, for the orientation determined by ${}_{(\alpha)}\lambda^i$ and ${}_{(\beta)}\lambda^i$, and the e 's are defined in a manner analogous to (5.2). It follows from (5.4), (5.6), and (6.1) that

$$(6.2) \quad e^{2\sigma}\bar{r}_{\alpha\beta} = r_{\alpha\beta} - e_\alpha e'_\alpha \frac{dk'_\alpha}{ds_\alpha} - e_\beta e'_\beta \frac{dk'_\beta}{ds_\beta} + e_\sigma k_\sigma^2,$$

where k'_α and k'_β are the first curvatures of C'_α and C'_β and where s_α and s_β are the arc lengths of C_α and C_β , respectively.

If \bar{V}_n is a space of constant curvature K_0 , the right-hand member of (6.2) does not depend upon the orientation determined by ${}_{(\alpha)}\lambda^i$ and ${}_{(\beta)}\lambda^i$ but is a scalar function in V_n . The algebraic sign of this function is constant and agrees with that of K_0 . Conversely if the right-hand member of (6.2) is a scalar function, it follows from (6.2) that \bar{V}_n has the same Riemann curvature for every orientation at P . By Schur's theorem the curvature of \bar{V}_n is a constant K_0 . The sign of K_0 is determined by the scalar function. Since by a magnification two spaces of constant positive (or negative) may be mapped on each other so that their geodesics correspond, the precise value of K_0 must be indeterminate. We state these results in the following theorem:

THEOREM 6.1. *The necessary and sufficient condition that a family of conformal geodesics in V_n be the images of the geodesics of an S_n is that*

$$r_{\alpha\beta} - e_\alpha e'_\alpha \frac{dk'_\alpha}{ds_\alpha} - e_\beta e'_\beta \frac{dk'_\beta}{ds_\beta} + e_\sigma k_\sigma^2$$

be a point function in V_n . The S_n has positive, zero, or negative Riemann curvature according as this function is greater than, equal to, or less than zero.

According to (2.6) and (4.7),

$$(6.3) \quad e_\alpha(n-2)\sigma_{ij}(\alpha)\lambda^i(\alpha)\lambda^j = e^{2\sigma}\bar{\gamma}_\alpha - \gamma_\alpha - \Delta_2\sigma - (n-2)\Delta_1\sigma,$$

where γ_α and $\bar{\gamma}_\alpha$ are the Ricci or mean curvatures for the direction $(\alpha)\lambda^i$ of V_n and \bar{V}_n , respectively. It follows from (5.6) and (6.3) that

$$(6.4) \quad (n-2)e_\alpha e'_\alpha \frac{dk'_\alpha}{ds_\alpha} + \gamma_\alpha = (n-2)e_\sigma k_\sigma^2 + e^{2\sigma}\bar{\gamma}_\alpha - \Delta_2\sigma - (n-2)\Delta_1\sigma.$$

If \bar{V}_n , ($n > 2$), is an Einstein space, $\bar{\gamma}_\alpha$ is a constant. Hence the left-hand member of (6.4) is a scalar. The converse is also true. This proves the next theorem:

THEOREM 6.2. *The necessary and sufficient condition that a family of conformal geodesics in a V_n of dimensionality $n > 2$ be the images of the geodesics of an E_n is that*

$$(n-2)e_\alpha e'_\alpha \frac{dk'_\alpha}{ds_\alpha} + \gamma_\alpha$$

be a point function in V_n .

In addition to this characteristic property, we easily obtain further necessary properties of the conformal images of the geodesics of an Einstein space.

THEOREM 6.3. *If a family of conformal geodesics in a V_n of dimensionality $n > 2$ are the images of the geodesics of an E_n , the H -directions of the family coincide with the principal Ricci directions of V_n which are not tangent to null vectors.*

This is an immediate consequence of (2.4) and Theorem 4.1. For \bar{V}_n is an Einstein space if and only if

$$(6.5) \quad \bar{R}_{ij} = a\bar{g}_{ij},$$

where a is a constant.

THEOREM 6.4. *Let E_n and \bar{E}_n be conformal Einstein spaces of dimensionality $n > 2$. Then the conformal images in E_n of the geodesics of \bar{E}_n as well as the images in \bar{E}_n of the geodesics of E_n are geodesic circles.*

It has been shown by Brinkmann* that a large class of Einstein spaces exist which are conformal to Einstein spaces. The above theorem applies to these spaces. The proof of the theorem follows. By definition of E_n ,

$$(6.6) \quad R_{ij} = bg_{ij},$$

* H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Mathematische Annalen, vol. 94 (1925), pp. 119-145.

where b is a constant. It follows from (2.4), (4.7), (6.5), and (6.6) that

$$\sigma_{ij} = \phi g_{ij},$$

where ϕ is a scalar. Hence every direction is an H -direction so that, as follows from (3.4), $dk_1/ds=0$, $k_2=0$ for each conformal geodesic. This proves Theorem 6.4. According to Theorem 4.1, the conformal geodesics of V_n and \bar{V}_n , ($n>2$), will be geodesic circles if and only if the mapping is such that

$$\bar{R}_{ij} - R_{ij} = \psi g_{ij},$$

where ψ is a scalar function.

7. Conformal geodesics in an S_n . The geometric property of conformal geodesics in a V_n stated in Theorem 2.2 is not intrinsic since it depends upon the tangent flat S_n . The only exception arises when V_n is itself a flat space. In this case, Theorem 2.2 becomes: The centers of curvature of the curves of a family of conformal geodesics which pass through a common point lie on a flat S_{n-1} orthogonal to the direction of $\sigma_{,i}$. In what follows, we show that the spaces of constant curvature enjoy an analogous property. The results apply without modification to all velocity systems.

We begin by generalizing the notion of center of curvature to apply to a curve C in V_n . Let V_2 be the osculating geodesic surface of C at P , and let C'' be any curve in V_2 which at P has the same tangent and principal normal* as C . The limiting first point of intersection (when it exists) of the geodesics of V_2 normal to C'' at P and at a nearby point Q as Q approaches P is called the *center of curvature of C in V_n* at the point P . From the viewpoint of the calculus of variations, the center of curvature of C is the focal point of C'' on the geodesic normal to C'' at P .† Since the focal point depends only on the first curvature of C'' at P , if it exists it is uniquely determined by C .

In accordance with Theorem 2.3, the osculating geodesic surfaces of the curves of a family of conformal geodesics $\{C\}$ which pass through a common point P all contain the gradient $\sigma_{,i}$. This means that a geodesic surface V_2 at P osculates ∞^1 conformal geodesics passing through P . The locus of the centers of curvature in V_n of these conformal geodesics is, in general, a curve in V_2 . In what follows, we prove the following theorem:

THEOREM 7.1. *Let $\{C\}$ be any family of conformal geodesics in a V_n whose first fundamental form is positive definite. Then the locus of the centers of curva-*

* Since V_2 is geodesic at P , C'' has the same first curvature at P when considered as a curve in V_2 or V_n .

† O. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 323, and M. Morse, *The Calculus of Variations in the Large*, American Mathematical Society Colloquium Publications, vol. 18, New York, 1934, pp. 51-55.

ture in V_n (if they exist) of the ∞^1 curves of $\{C\}$ which pass through a common point osculating the same geodesic surface V_2 at that point is a geodesic of V_2 if and only if V_2 is an S_2 .

The existence of the center of curvature of a curve in S_n is discussed later. If we use the geodesic polar coordinates with center at P , the first fundamental form of V_2 is

$$(7.1) \quad ds^2 = dr^2 + G(r, \theta)d\theta^2,$$

where

$$(7.2) \quad (G(0, \theta))^{1/2} = 0, \quad \frac{\partial(G(0, \theta))^{1/2}}{\partial r} = 1.$$

Now the center of curvature in V_n of a curve osculating V_2 at P depends only on its direction and first curvature k_1 at P . If (r, θ) are the coordinates of the center of curvature, it follows that

$$(7.3) \quad k_1 = f(r, \theta).$$

The function $f(r, \theta)$ is completely determined by the surface V_2 . Indeed, it may be shown by the methods of the calculus of variations that

$$(7.4) \quad f(r, \theta) = - \int \frac{dr}{G(r, \theta)}, \quad \lim_{r \rightarrow 0} \left[\int \frac{dr}{G(r, \theta)} + \frac{1}{r} \right] = 0.$$

This relation is not used in the present proof.

According to (3.5), the first curvatures of the ∞^1 curves of $\{C\}$ osculating V_2 at P obey an equation of the form

$$(7.5) \quad k_1 = a \sin(\theta + b),$$

where a and b are constants which depend upon the particular family $\{C\}$ and the point P .^{*} From (7.3) and (7.5), the locus of the centers of curvature of the ∞^1 conformal geodesics is

$$(7.6) \quad f(r, \theta) = a \sin(\theta + b).$$

By hypothesis this locus is a geodesic of V_2 for every value of a and b . Differentiating (7.6), we find that the curves (7.6) are the solutions of

$$(7.7) \quad \frac{d^2 r}{d\theta^2} + \frac{f_{rr}}{f_r} \left(\frac{dr}{d\theta} \right)^2 + \frac{2f_{r\theta}}{f_r} \left(\frac{dr}{d\theta} \right) + \frac{f_{\theta\theta} + f}{f_r} = 0.$$

But (7.6) must also satisfy the differential equation for the geodesics of V_2 :

^{*} Indeed, for the family $\{C\}$ associated with the conformal mapping (2.3), $a = (\Delta_1 \sigma)^{1/2}$ and b is chosen so that the vector $\sigma_{,1}$ is tangent to $\theta = -b$.

$$(7.8) \quad \frac{d^2 r}{d\theta^2} - \frac{G_r}{G} \left(\frac{dr}{d\theta} \right)^2 - \frac{G_\theta}{2G} \left(\frac{dr}{d\theta} \right) - \frac{G_r}{2} = 0.$$

It follows from (7.7) and (7.8) that each curve (7.6) will be a geodesic if and only if

$$(7.9) \quad \frac{f_{rr}}{f_r} + \frac{G_r}{G} = 0, \quad \frac{2f_{\theta r}}{f_r} + \frac{G_\theta}{2G} = 0, \quad \frac{f_{\theta\theta} + f}{f_r} + \frac{G_r}{2} = 0.$$

A little calculation shows that the solution of (7.2) and (7.9) is (7.4) and one of the following equations:

$$(7.10) \quad G(r, \theta) = \frac{1}{c^2} \sin^2 cr, \quad G(r, \theta) = r^2, \quad G(r, \theta) = \frac{1}{c^2} \sinh^2 cr.$$

Since the Gaussian curvature of V_2 is equal to $-(G^{1/2})_{rr}/G^{1/2}$, it follows from (7.10) that V_2 is a surface of constant curvature c^2 , 0, or $-c^2$, respectively. This completes the proof.

If V_n is an S_n , it follows easily from (7.4), (7.5), and (7.10) that the locus of centers of curvature in any geodesic S_2 is*

$$(7.11) \quad c \cot cr = (\Delta_1 \sigma)^{1/2} \sin \theta, \quad \frac{1}{r} = (\Delta_1 \sigma)^{1/2} \sin \theta, \quad c \coth cr = (\Delta_1 \sigma)^{1/2} \sin \theta,$$

according as S_n has Riemann curvature c^2 , 0, or $-c^2$, respectively. Hence the locus always exists in an S_n of positive or zero curvature and exists in an S_n of negative Riemann curvature $-c^2$ if and only if $\Delta_1 \sigma \sin^2 \theta > c^2$.

In an S_n , every geodesic V_2 is a totally geodesic S_2 . Therefore, the locus of the centers of curvature in any S_2 of the appropriate curves passing through P of a family of conformal geodesics $\{C\}$ is a geodesic of S_n . This geodesic is easily shown to be orthogonal to the geodesic of S_n which is tangent to $\sigma_{,i}$ at P . Furthermore, the point at which the two geodesics intersect orthogonally does not depend upon the particular osculating geodesic surface at P . Hence the totality of geodesics in S_n which are the loci associated with the family $\{C\}$ lie on a totally geodesic S_{n-1} orthogonal to the geodesic which is tangent to $\sigma_{,i}$. We state this result in the following theorem:

THEOREM 7.2. *Let $\{C\}$ be a family of conformal geodesics in an S_n whose first fundamental form is positive definite. Then the centers of curvature in S_n (if they exist) of the curves of $\{C\}$ which pass through a common point lie on a totally geodesic S_{n-1} orthogonal to the geodesic through the point which is tangent to the fixed vector of Theorem 2.1.*

* The direction of the gradient $\sigma_{,i}$ is $\theta=0$.

III. SUBSPACES OF V_n

8. **Conformal geodesics in subspaces of V_n .** The conformal transformation (2.3) induces a conformal mapping of the respective subspaces of V_n and \bar{V}_n upon each other. If V_p and \bar{V}_p , ($1 < p < n$), are two such conformal subspaces, the images of the geodesics of \bar{V}_p in V_p are conformal geodesics of V_p and therefore enjoy the properties of conformal geodesics derived in the previous sections. We now consider the additional relationships which exist between the families of conformal geodesics of V_n and of its subspaces which are the conformal images of the geodesics of \bar{V}_n and its respective subspaces under the transformation (2.3). We refer to these conformal geodesics of V_n and of its subspaces as *corresponding* families of conformal geodesics. Any one such family is said to *correspond* to the transformation (2.3). It is clear that if a family of conformal geodesics of V_n is given, then the conformal transformation is determined except for a magnification. Hence all the corresponding families of conformal geodesics in the subspaces of V_n are uniquely determined by the given conformal geodesics of V_n .

The equations of the imbedding of a V_p in V_n are* $x^i = x^i(y^\alpha)$. If the first fundamental form of V_n is (2.1), the corresponding form of V_p is

$$(8.1) \quad ds^2 = h_{\alpha\beta} dy^\alpha dy^\beta,$$

where†

$$(8.2) \quad h_{\alpha\beta} = g_{ij} x^i_{,\alpha} x^j_{,\beta}.$$

If ζ^α is a unit vector in V_p , the principal normal μ^i of the conformal geodesic of V_n tangent to ζ^α is given by (2.8), where ξ^i are the components in the x 's of ζ^α so that

$$(8.3) \quad \xi^i = \zeta^\alpha x^i_{,\alpha}.$$

The principal normal ν^α of the corresponding conformal geodesic of V_p tangent to ζ^α is

$$\nu^\alpha = e\sigma_{,\beta}(h^{\alpha\beta} - e\zeta^\alpha\zeta^\beta),$$

where $\sigma_{,\beta}$ given by

$$(8.4) \quad \sigma_{,\beta} = \sigma_{,m} x^m_{,\beta}$$

is the projection of $\sigma_{,i}$ in the tangent vector space of V_p . If η^i are the components of ν^α in the x 's, $\eta^i = \nu^\alpha x^i_{,\alpha}$, so that

* In this section, the range of the Greek letters is 1, 2, \dots , p unless otherwise stated.

† The comma followed by a Greek letter denotes covariant differentiation with respect to the form (8.1) and the y 's.

$$(8.5) \quad \eta^i = e\sigma_{,m}x_{,\beta}^m x_{,\alpha}^i (h^{\alpha\beta} - e\xi^\alpha \xi^\beta).$$

It is easy to show from the canonical representation for $h^{\alpha\beta}$ and g^{ij} that

$$(8.6) \quad h^{\alpha\beta} x_{,\alpha}^i x_{,\beta}^m = g^{im} - \sum_{\tau=p+1}^n e_\tau ({}_{(\tau)}\lambda^i ({}_{(\tau)}\lambda^m),$$

where the $({}_{(\tau)}\lambda^i$ are $n-p$ mutually orthogonal unit normals to V_p and $e_\tau = g_{ij} ({}_{(\tau)}\lambda^i ({}_{(\tau)}\lambda^j$. It follows from (2.8), (8.3), (8.5), and (8.6) that*

$$(8.7) \quad \eta^i = \mu^i - e\sigma_{,m} \sum_\tau e_\tau ({}_{(\tau)}\lambda^m ({}_{(\tau)}\lambda^i).$$

It is clear that the last expression in (8.7) is the signed projection of $\sigma_{,m}$ in the normal vector space of V_p . As an immediate consequence of (8.7), it is seen that *if two V_p 's are tangent at a point, the principal normals of their corresponding conformal geodesics which pass through the point in the same direction are equal.*

While the principal normal is thus determined by the tangent vector space of V_p , the H -directions also depend upon the tensors $({}_{(\tau)}\Omega_{\alpha\beta}$. These tensors are introduced in the equations†

$$(8.8) \quad x_{,\alpha\beta}^i = -\{i|jk\}x_{,\alpha}^j x_{,\beta}^k + \sum_\tau e_\tau ({}_{(\tau)}\Omega_{\alpha\beta} ({}_{(\tau)}\lambda^i$$

and are used to construct the second fundamental form of V_p . According to Theorem 3.1, the H -directions of the corresponding family of conformal geodesics of V_p are the non-null principal directions determined by the tensor $\sigma_{\alpha\beta}$ where

$$(8.9) \quad \sigma_{\alpha\beta} = \sigma_{,\alpha\beta} - \sigma_{,\alpha}\sigma_{,\beta}.$$

By straightforward calculation, we find from (4.1), (8.4), (8.8), and (8.9) that

$$(8.10) \quad \sigma_{\alpha\beta} = \sigma_{ij}x_{,\alpha}^i x_{,\beta}^j + \sum_\tau e_\tau ({}_{(\tau)}\lambda^i \sigma_{,i} ({}_{(\tau)}\Omega_{\alpha\beta}.$$

If ξ^i is an H -direction at a point P of a family of conformal geodesics of V_n , in accordance with Theorem 3.1,

$$(8.11) \quad (\sigma_{ij} - \rho g_{ij})\xi^i = 0,$$

where $e\rho = \sigma_{ij}\xi^i\xi^j$ and $e = g_{ij}\xi^i\xi^j$. For any V_p which contains ξ^i , the components ξ^α in the y 's of ξ^i satisfy (8.3). We multiply (8.11) by $x_{,\beta}^j$ and sum the

* This equation may also be obtained directly from Theorem 2.1.

† Eisenhart, loc. cit., p. 160.

resulting equation for j . After using (8.2) and (8.3), this equation becomes

$$(\sigma_{ij}x_{,\alpha}^i x_{,\beta}^j - \rho h_{\alpha\beta})\zeta^\alpha = 0.$$

Hence ζ^α is a principal direction determined by the tensor $\sigma_{ij}x_{,\alpha}^i x_{,\beta}^j$. But if V_p is geodesic or umbilical at a point P , it follows from (8.10) that the H -directions at P of the corresponding family of conformal geodesics of V_p are determined by $\sigma_{ij}x_{,\alpha}^i x_{,\beta}^j$. This discussion proves the following theorem:

THEOREM 8.1. *If at a point an H -direction of a family of conformal geodesics of V_n is tangent to a subspace V_p which is geodesic or umbilical at the point, then this direction is also an H -direction of the corresponding family of conformal geodesics of V_p .*

We now suppose $p = n - 1$, and write (8.10) as

$$(8.12) \quad \sigma_{\alpha\beta} = \sigma_{ij}x_{,\alpha}^i x_{,\beta}^j + e_1 (1)\lambda^i \sigma_{,i} \Omega_{\alpha\beta}.$$

As was shown in the paragraph below (4.3), if ξ^i is a principal direction determined by two of the tensors $\sigma_{\alpha\beta}$, $\sigma_{ij}x_{,\alpha}^i x_{,\beta}^j$, $\Omega_{\alpha\beta}$, it is also determined by the third provided $(1)\lambda^i \sigma_{,i} \neq 0$. But $\Omega_{\alpha\beta}$ determines the directions of the lines of curvature of V_{n-1} , and, as was shown above, $\sigma_{ij}x_{,\alpha}^i x_{,\beta}^j$ determines the H -directions of the geodesic V_{n-1} which has the same orientation as V_{n-1} . This proves the next theorem:

THEOREM 8.2. *Let V_{n-1} be a hypersurface of V_n which at a point does not contain the fixed vector of Theorem 2.1. Then if, at this point, a vector is a member of two of the following sets, it is also a member of the third set:*

- (1) *the tangents of the lines of curvature of V_{n-1} ,*
- (2) *the H -directions of a family of conformal geodesics of V_{n-1} ,*
- (3) *the H -directions of the corresponding family of conformal geodesics of the tangent geodesic V_{n-1} .*

As a consequence of Theorem 8.1 and Theorem 8.2, we note that an H -direction of a family of conformal geodesics of V_n is also an H -direction of the corresponding family of conformal geodesics of a hypersurface V_{n-1} with $(1)\lambda^i \sigma_{,i} \neq 0$ if and only if the direction is tangent to a line of curvature of V_{n-1} .

Now the corresponding family of conformal geodesics of a hypersurface $\sigma = \text{const.}$ is simply the totality of geodesics of the hypersurface; so the H -directions of this family are completely indeterminate. It follows from the statement italicized above that an H -direction of a family of conformal geodesics of V_n which lies in a hypersurface $\sigma = \text{const.}$ with $\Delta_1 \sigma \neq 0$ is tangent to a line of curvature of the hypersurface.

We now consider two hypersurfaces V_{n-1} and V_{n-1}^* which are tangent at a point P of V_n . If their equations of imbedding in V_n are $x^i = x^i(y^\alpha)$ and $x^i = x^{*i}(y^\alpha)$, it follows that at P , $\partial x^i / \partial y^\alpha$ and $\partial x^{*i} / \partial y^\alpha$ span the same tangent vector space. Hence we may choose the coordinate directions y^α for V_{n-1} and V_{n-1}^* as mutually tangent at P so that at this point

$$(8.13) \quad \frac{\partial x^i}{\partial y^\alpha} = \frac{\partial x^{*i}}{\partial y^\alpha}.$$

Now the tensor $\sigma_{\alpha\beta}$ for V_{n-1} is given by (8.12), and the corresponding tensor $\sigma_{\alpha\beta}^*$ for V_{n-1}^* by

$$(8.14) \quad \sigma_{\alpha\beta}^* = \sigma_{ij} x_{,\alpha}^{*i} x_{,\beta}^{*j} + e_1^* ({}_1\lambda^{*i} \sigma_{,i} \Omega_{\alpha\beta}^*),$$

where the notation is analogous to that used in (8.12) and refers to V_{n-1}^* . Since at P , $({}_1\lambda^i = {}_1\lambda^{*i}$, it follows from (8.12), (8.13), and (8.14) that at the point of contact,

$$(8.15) \quad \sigma_{\alpha\beta} - \sigma_{\alpha\beta}^* = e_1 ({}_1\lambda^i \sigma_{,i} (\Omega_{\alpha\beta} - \Omega_{\alpha\beta}^*)).$$

If $({}_1\lambda^i \sigma_{,i} = 0$, it follows that $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^*$. More generally, if two V_p 's are tangent at P and contain the gradient $\sigma_{,i}$ at P , the H -directions of the corresponding families of conformal geodesics coincide at this point.

If $({}_1\lambda^i \sigma_{,i} \neq 0$ and ζ^α denotes a unit vector of V_{n-1} and V_{n-1}^* at P , we obtain

$$(8.16) \quad \sigma_{\alpha\beta} \zeta^\alpha \zeta^\beta - \sigma_{\alpha\beta}^* \zeta^\alpha \zeta^\beta = e_1 ({}_1\lambda^i \sigma_{,i} (\Omega_{\alpha\beta} \zeta^\alpha \zeta^\beta - \Omega_{\alpha\beta}^* \zeta^\alpha \zeta^\beta)).$$

According to (5.6),

$$(8.17) \quad \sigma_{\alpha\beta} \zeta^\alpha \zeta^\beta = e' \frac{dk'}{ds} - e e_\sigma k_\sigma^2, \quad \sigma_{\alpha\beta}^* \zeta^\alpha \zeta^\beta = e' \frac{dk^{*'}}{ds^*} - e e_\sigma k_\sigma^{*2},$$

where the notation is analogous to that of (5.6) and refers to the corresponding families of conformal geodesics of V_{n-1} and V_{n-1}^* . The remarks following (8.7) show that

$$(8.18) \quad k_\sigma = k_\sigma^*.$$

Of course

$$(8.19) \quad \Omega_{\alpha\beta} \zeta^\alpha \zeta^\beta = eK, \quad \Omega_{\alpha\beta}^* \zeta^\alpha \zeta^\beta = eK^*,$$

where K and K^* denote the normal curvatures of V_{n-1} and V_{n-1}^* , respectively, for the direction ζ^α . It follows from (8.16), (8.17), (8.18), and (8.19) that†

† A similar equation may be obtained for a single V_{n-1} by using (8.12) instead of (8.15) in the above derivation.

$$(8.20) \quad \frac{dk'}{ds} - \frac{dk^{*'}}{ds^*} = e_1 e e'_{(1)} \lambda^i \sigma_{,i} (K - K^*).$$

Hence the difference of normal curvatures for the same direction on two tangent hypersurfaces is expressible in terms of the curvatures of the corresponding conformal geodesics of these hypersurfaces. It also follows from (8.20) that the expression

$$\left(\frac{dk'}{ds} - \frac{dk^{*'}}{ds^*} \right) \frac{1}{(1) \lambda^i \sigma_{,i}}$$

does not depend upon the conformal mapping (2.3) of V_n upon \bar{V}_n ; that is, it is invariant for any pair of corresponding families of conformal geodesics.

As an immediate consequence of (8.15) and the remarks below (4.3), we have the following theorem:

THEOREM 8.3. *Let V_{n-1} and V_{n-1}^* be tangent at a point where they do not contain the fixed vector of Theorem 2.1. Then if, at this point, a vector is a member of three of the following sets, it is also a member of the fourth set:*

- (1) *the tangents of the lines of curvature of V_{n-1} ,*
- (2) *the tangents of the lines of curvature of V_{n-1}^* ,*
- (3) *the H -directions of a family of conformal geodesics of V_{n-1} ,*
- (4) *the H -directions of the corresponding family of conformal geodesics of V_{n-1}^* .*

If the difference of the normal curvatures at P of V_{n-1} and V_{n-1}^* for the same direction is constant as the direction changes, it follows that $\Omega_{\alpha\beta} = \Omega_{\alpha\beta}^* + a h_{\alpha\beta}$, where a is a constant. In this case, according to (8.15), the H -directions for any corresponding families of conformal geodesics in V_{n-1} and V_{n-1}^* coincide at P .

A conformal transformation of V_n for which

$$(8.21) \quad \sigma_{ij} = \phi g_{ij}$$

has a particularly simple character. As noted in §6, it is only in this case that the corresponding conformal geodesics of V_n are geodesic circles. We investigate the induced conformal transformations of the hypersurfaces of V_n . From (8.2), (8.12), and (8.21),

$$(8.22) \quad \sigma_{\alpha\beta} = \phi h_{\alpha\beta} + e_1 (1) \lambda^i \sigma_{,i} \Omega_{\alpha\beta}$$

for any V_{n-1} in V_n . Hence if $(1) \lambda^i \sigma_{,i} = 0$ at a point, every direction is an H -direction of the corresponding conformal geodesics of V_{n-1} at this point. If $(1) \lambda^i \sigma_{,i} \neq 0$, it follows from (8.22) that the non-null tangents to the lines of

curvature of V_{n-1} and the H -directions of the corresponding conformal geodesics of V_{n-1} coincide.

Conversely, suppose the direction of each line of curvature of any V_{n-1} in V_n is an H -direction of a family of conformal geodesics of V_{n-1} if the direction is not tangent to a null vector, and suppose that all of these families of conformal geodesics correspond to the same conformal mapping of V_n . Now it is easy to show* that a V_{n-1} in V_n exists which contains an arbitrary point P of V_n and is such that the lines of curvature of V_{n-1} are tangent to an arbitrary ennuple of non-null directions at P . Furthermore, we may choose coordinates y^α in the V_{n-1} so that the tangents $x'_{,\alpha}$ (α constant) to the coordinate lines are also tangent to the lines of curvature at P . In this coordinate system

$$\sigma_{\alpha\beta} = 0, \quad \Omega_{\alpha\beta} = 0, \quad \alpha \neq \beta.$$

It follows from (8.12) that $\sigma_{ij}x'^i_{,\alpha}x'^j_{,\beta} = 0$, ($\alpha \neq \beta$). Since $x'^i_{,\alpha}$ and $x'^j_{,\beta}$ are arbitrary orthogonal vectors in V_n , the last equation shows that $\sigma_{ij} = \phi g_{ij}$. This proves the next theorem:

THEOREM 8.4. *Let V_n be conformal to \bar{V}_n so that the images of the geodesics of \bar{V}_n are geodesic circles in V_n . Then and only then the non-null tangents to the lines of curvature of any V_{n-1} in V_n are H -directions for the corresponding family of conformal geodesics in V_{n-1} .*

It is easy to see that non-trivial conformal transformations exist for which (8.21) holds. As noted in §6, the conformal mapping of any two Einstein spaces of dimensionality $n > 2$ gives rise to an equation of the form (8.21). We discuss this topic further in §12.

As a consequence of Theorem 6.4 and the remarks following (8.22), we have the following theorem which may be illustrated by non-trivial examples:

THEOREM 8.5. *Let E_n and \bar{E}_n be conformal Einstein spaces of dimensionality $n > 3$, and let E_{n-1} and \bar{E}_{n-1} be Einstein hypersurfaces which correspond by the mapping and which do not contain the fixed vector of Theorem 2.1. Then E_{n-1} and \bar{E}_{n-1} have indeterminate lines of curvature.*

9. The hypersurfaces $\sigma = \text{const.}$ For the conformal transformation (2.3), the hypersurfaces $\sigma = \text{const.}$ play a special role. The mapping of these hypersurfaces in V_n and \bar{V}_n , respectively, upon each other is simply a change in scale. We investigate the conditions under which the normal to $\sigma = \text{const.}$ may be an H -direction of the corresponding family of conformal geodesics of V_n . We obtain the following results:

* For a proof of this statement, cf. A. Fialkow, *The Riemannian curvature of a hypersurface*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 256-257.

THEOREM 9.1. *The tangent at a point of V_n to a curve of the congruence normal to the hypersurfaces $\sigma = \text{const.}$ is an H -direction of the corresponding family of conformal geodesics of V_n if and only if the curve is not tangent to a null vector and has zero first curvature at the point.*

The proof follows. According to Theorem 3.1 and the hypothesis that $g^{im}\sigma_{,m}$ is an H -direction at a point P of V_n , $g^{im}\sigma_{,m}$ is a non-null principal direction determined by the tensor (4.1) at P . Hence, for a suitable ρ ,

$$(9.1) \quad (\sigma_{,ij} - \sigma_{,i}\sigma_{,j} - \rho g_{ij})\sigma^{,i} = 0$$

at this point, where $\sigma^{,i} = g^{im}\sigma_{,m}$. Let* ${}_{(p)}\lambda^i$ be $n-1$ mutually orthogonal congruences of vectors in V_n such that

$$(9.2) \quad \sigma_{,i} {}_{(p)}\lambda^i = 0.$$

Differentiating (9.2) covariantly, we obtain

$$\sigma_{,ij} {}_{(p)}\lambda^j + \sigma_{,i} {}_{(p)}\lambda^j{}_{,i} = 0.$$

Hence

$$(9.3) \quad -\sigma_{,ij} {}_{(p)}\lambda^j \sigma^{,i} = {}_{(p)}\lambda_{j,i} \sigma^{,j} \sigma^{,i}.$$

But from (9.1) and (9.2), $\sigma_{,ij} {}_{(p)}\lambda^j \sigma^{,i} = 0$. If we substitute this value in (9.3),

$$(9.4) \quad {}_{(p)}\lambda_{j,i} {}_{(n)}\lambda^i {}_{(n)}\lambda^j = 0,$$

where ${}_{(n)}\lambda^i$ is a unit vector tangent to $\sigma^{,i}$. It is known† that (9.4) is the condition that a curve of the congruence whose tangents are ${}_{(n)}\lambda^i$ have zero first curvature. This proves one of the statements in the theorem. The converse may be demonstrated by reversing the steps of the above proof.

We now show that under the conditions of the hypothesis of Theorem 9.1, the non-null directions of the lines of curvature of the corresponding hypersurface $\sigma = \text{const.}$ are also H -directions at P . By a change of coordinates we may write

$$(9.5) \quad \sigma = x^n, \quad g_{nn} = \frac{1}{g^{nn}} = e_n H^2(x^i), \quad g_{np} = 0.$$

From (9.5), we find

$$(9.6) \quad \sigma_{,p} = 0, \quad \sigma_{,n} = 1,$$

$$(9.7) \quad \sigma_{,np} = -\frac{H_{,p}}{H}.$$

* In this section the indices p, q have the range $1, 2, \dots, n-1$.

† Eisenhart, loc. cit., p. 100.

Since the first curvature of the curve

$$(9.8) \quad x^p = \text{const.}, \quad x^n = x^n$$

is zero at P ,

$$(9.9) \quad \frac{d^2 x^i}{ds^2} + \{i|jk\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

at this point. According to (9.8),

$$(9.10) \quad \frac{dx^n}{ds} = \frac{1}{H}, \quad \frac{dx^p}{ds} = 0.$$

From (9.5), (9.10), and (9.9) with $i=q$, we obtain $g^{pq}H_{,p}=0$. Since the rank of $\|g^{pq}\|$ is $n-1$, this last equation is equivalent to $H_{,p}=0$ at P . Hence, from (9.7), $\sigma_{,np}=0$. As a consequence of this equation and (9.6),

$$(9.11) \quad \sigma_{np} = 0$$

at the point P .

The principal directions determined by σ_{ij} are the vectors λ^i such that $(\sigma_{ij} - \rho g_{ij})\lambda^i = 0$ for suitable ρ . It follows from (9.5) and (9.11) that the coordinate direction whose parameter is x^n is a principal direction determined by σ_{ij} and the vectors orthogonal to it which satisfy the equation

$$(9.12) \quad (\sigma_{pq} - \rho g_{pq})\lambda^p = 0$$

are also H -directions. Since the tensor $\sigma_{\alpha\beta}$ defined by (8.9) for the hypersurface

$$x^\alpha = y^\alpha, \quad x^n = \text{const.}$$

is identically zero, (8.12) becomes, after a slight change in notation,

$$(9.13) \quad \sigma_{ij}\delta_\alpha^i\delta_\beta^j + e_n(e_n\Delta_1\sigma)^{1/2}\Omega_{\alpha\beta} = 0.$$

Furthermore, the tensors $\sigma_{ij}\delta_\alpha^i\delta_\beta^j$ and $h_{\alpha\beta}$ are, respectively, equal to σ_{pq} and g_{pq} after a change of notation. Since $\Delta_1\sigma \neq 0$, it follows from (9.12) and (9.13) that the principal directions determined by σ_{ij} at P other than the normal to $\sigma = \text{const.}$ are also determined by $\Omega_{\alpha\beta}$. Hence they are the directions of the lines of curvature of $\sigma = \text{const.}$ at P .

Conversely, suppose that the directions of the lines of curvature at P of $\sigma = \text{const.}$ are all H -directions of the corresponding family of conformal geodesics of V_n . Then none of the lines of curvature at P are tangent to null vectors. In accordance with the theory of principal directions determined by the tensor $\Omega_{\alpha\beta}$, it follows that $n-1$ mutually orthogonal vectors ${}_{(p)}\lambda^i$ exist

at P which are tangent to lines of curvature. Hence for a proper value of ρ_p

$$(9.14) \quad (\sigma_{ij} - \rho_p g_{ij})_{(p)} \lambda^i = 0.$$

If $_{(n)}\lambda^i$ is a unit vector normal to $\sigma = \text{const.}$ at P and

$$e_n = g_{ij} \, _{(n)}\lambda^i \, _{(n)}\lambda^j, \quad e_n \rho_n = \sigma_{ij} \, _{(n)}\lambda^i \, _{(n)}\lambda^j,$$

it follows from (9.14) that

$$(9.15) \quad (\sigma_{ij} - \rho_n g_{ij}) \, _{(n)}\lambda^i \, _{(p)}\lambda^j = 0, \quad (\sigma_{ij} - \rho_n g_{ij}) \, _{(n)}\lambda^i \, _{(n)}\lambda^j = 0.$$

Since the vectors $_{(p)}\lambda^i, \, _{(n)}\lambda^i$ are n mutually orthogonal vectors, it follows from (9.15) that $(\sigma_{ij} - \rho_n g_{ij}) \, _{(n)}\lambda^i = 0$; so the normal at P is also an H -direction of the corresponding family of conformal geodesics of V_n . This proves the following theorem:

THEOREM 9.2. *If the normal at a point to a hypersurface $\sigma = \text{const.}$ is an H -direction of the corresponding family of conformal geodesics of V_n , the non-null tangents to the lines of curvature of the hypersurface at the point are also H -directions. Conversely, if at a point all the tangents to the lines of curvature of $\sigma = \text{const.}$ are H -directions and the normal to the hypersurface at this point is a non-null vector, then the normal is also an H -direction.*

In particular, the above two theorems are true at every point of V_n only if the hypersurfaces $\sigma = \text{const.}$ are parallel. We note that if a one-parameter family of hypersurfaces in V_n is parallel and its image in \bar{V}_n is also parallel, the family consists of the hypersurfaces $\sigma = \text{const.}$ Since $\Delta_1 \sigma = f(\sigma)$ if the hypersurfaces $\sigma = \text{const.}$ are parallel, in accordance with (5.4) it is characteristic in this case that k_σ has a constant value on each of these hypersurfaces.

IV. SOME SPECIAL QUESTIONS

10. The Frenet equations and conformal geodesics. The last two equations of (3.6) are equivalent to

$$(10.1) \quad \frac{d\tau_i}{ds} \, _{(3)}\xi^i = e_1 e_2 k_1 k_2, \quad \frac{d\tau_i}{ds} \, _{(1+r)}\xi^i = 0, \quad r > 2,$$

where the notation is that used in §3. We show by mathematical induction that the following equations hold for any velocity system (2.9) (in particular, for any family of conformal geodesics):

$$(10.2) \quad \begin{aligned} \frac{d^p \tau_i}{ds^p} \, _{(p+2)}\xi^i &= e_1 e_2 \cdots e_{p+1} k_1 k_2 \cdots k_{p+1}, \\ \frac{d^p \tau_i}{ds^p} \, _{(p+r)}\xi^i &= 0, \quad r > 2; p = 0, 1, \cdots, n-2. \end{aligned}$$

From (3.5) and (10.1), it follows that (10.2) holds for $p=0, 1$. We assume that (10.2) holds for $p=0, 1, \dots, m$. Since $(\mathfrak{d}^m \tau_i / \mathfrak{d}s^m)_{(m+3)} \xi^i = 0$, we find by covariant differentiation with respect to s and use of (3.1) that

$$(10.3) \quad \frac{\mathfrak{d}^{m+1} \tau_i}{\mathfrak{d}s^{m+1}}_{(m+3)} \xi^i + \frac{\mathfrak{d}^m \tau_i}{\mathfrak{d}s^m} [-e_{m+2} k_{m+2}{}_{(m+2)} \xi^i + e_{m+4} k_{m+3}{}_{(m+4)} \xi^i] = 0.$$

From (10.3) and (10.2) with $p=m$,

$$\frac{\mathfrak{d}^{m+1} \tau_i}{\mathfrak{d}s^{m+1}}_{(m+3)} \xi^i = e_1 e_2 \cdots e_{m+2} k_1 k_2 \cdots k_{m+2},$$

which is the first equation of (10.2) with $p=m+1$. Similarly, if we differentiate

$$\frac{\mathfrak{d}^m \tau_i}{\mathfrak{d}s^m}{}_{(m+1+r)} \xi^i = 0, \quad r > 2,$$

covariantly with respect to s and use (3.1), we obtain

$$\frac{\mathfrak{d}^{m+1} \tau_i}{\mathfrak{d}s^{m+1}}{}_{(m+1+r)} \xi^i + \frac{\mathfrak{d}^m \tau_i}{\mathfrak{d}s^m} [-e_{m+r} k_{m+r}{}_{(m+r)} \xi^i + e_{m+r+2} k_{m+r+1}{}_{(m+r+2)} \xi^i] = 0.$$

According to (10.2) with $p=m$, this equation becomes

$$\frac{\mathfrak{d}^{m+1} \tau_i}{\mathfrak{d}s^m}{}_{(m+1+r)} \xi^i = 0$$

which is the second equation of (10.2) with $p=m+1$. This completes the induction.

We apply these equations to the conformal geodesics of V_n and \bar{V}_n which are the images of the geodesics of \bar{V}_n and V_n , respectively, under the conformal transformation (2.3). Then (10.2) with $p=0, 1$ becomes

$$(10.4) \quad \sigma_{,i}{}_{(2)} \xi^i = e_1 k_1, \quad \sigma_{,i}{}_{(r)} \xi^i = 0,$$

$$(10.5) \quad \frac{\mathfrak{d} \sigma_{,i}}{\mathfrak{d}s}{}_{(3)} \xi^i = e_1 e_2 k_1 k_2, \quad \frac{\mathfrak{d} \sigma_{,i}}{\mathfrak{d}s}{}_{(1+r)} \xi^i = 0, \quad r > 2,$$

for V_n and

$$(10.6) \quad (-\sigma)_{,i}{}_{(2)} \bar{\xi}^i = \bar{e}_1 \bar{k}_1, \quad (-\sigma)_{,i}{}_{(r)} \bar{\xi}^i = 0,$$

$$(10.7) \quad \frac{\mathfrak{d}(-\sigma)_{,i}}{\mathfrak{d}\bar{s}}{}_{(3)} \bar{\xi}^i = \bar{e}_1 \bar{e}_2 \bar{k}_1 \bar{k}_2, \quad \frac{\mathfrak{d}(-\sigma)_{,i}}{\mathfrak{d}\bar{s}}{}_{(1+r)} \bar{\xi}^i = 0, \quad r > 2,$$

for \bar{V}_n , where a notation analogous to that for V_n is used. If ${}_{(1)} \bar{\xi}^i$ is chosen as corresponding to ${}_{(1)} \xi^i$ at a point P ,

$$(10.8) \quad (1)\bar{\xi}^i = e^{-\sigma} (1)\xi^i,$$

and it follows from (3.3), (10.4), and (10.6) that at P ,

$$(2)\bar{\xi}^i = e^{-\sigma} (2)\xi^i, \quad \bar{k}_1 = -e^{-\sigma} k_1.$$

From (2.5) and (10.8)

$$(10.9) \quad \frac{\partial(-\sigma)_{,i}}{\partial s} = -e^{-\sigma} \left[\frac{\partial \sigma_{,i}}{\partial s} - 2 \frac{d\sigma}{ds} \sigma_{,i} + \Delta_1 \sigma (1)\xi_i \right].$$

It follows from (3.3), (10.5), (10.7), and (10.9) that at P

$$(3)\bar{\xi}^i = e^{-\sigma} (3)\xi^i, \quad \bar{k}_2 = e^{-\sigma} k_2.$$

11. Similar families of conformal geodesics. The families of conformal geodesics in V_n which correspond to the transformations (2.3) and

$$(11.1) \quad ds' = e^{f(\sigma)} ds, \quad f'(\sigma) \neq 0,$$

where $f'(\sigma) = df/d\sigma$ are called *similar* families of conformal geodesics. We denote these families by $\{C\}$ and $\{C\}'$, respectively. The equations analogous to (10.4) and (10.5) obtaining for $\{C\}'$ are

$$(11.2) \quad \begin{aligned} [f(\sigma)]_{,i} (2)\xi'^i &= e_1' k_1', & [f(\sigma)]_{,i} (r)\xi'^i &= 0, \\ \frac{\partial [f(\sigma)]_{,i}}{\partial s} (3)\xi'^i &= e_1' e_2' k_1' k_2', & \frac{\partial [f(\sigma)]_{,i}}{\partial s} (1+r)\xi'^i &= 0, \quad r > 2. \end{aligned}$$

The notation in these equations is analogous to that employed in (10.4) and (10.5), the prime referring to $\{C\}'$. We consider curves of the two families which are tangent at a point so that $(1)\xi'^i = (1)\xi^i$. Since

$$[f(\sigma)]_{,i} = f'(\sigma) \sigma_{,i}, \quad \frac{\partial [f(\sigma)]_{,i}}{\partial s} = f''(\sigma) \frac{d\sigma}{ds} \sigma_{,i} + f'(\sigma) \frac{\partial \sigma_{,i}}{\partial s},$$

it follows from (3.3), (10.4), (10.5), and (11.2) that

$$(2)\xi'^i = (2)\xi^i, \quad (3)\xi'^i = (3)\xi^i, \quad k_1' = f'(\sigma) k_1, \quad k_2' = k_2$$

at the point. Hence the ratio of the first curvatures of tangent curves of $\{C\}$ and $\{C\}'$ at a point is independent of their common initial direction. We also have the following theorem:

THEOREM 11.1. *All similar conformal geodesics which are tangent at a point of V_n have the same first and second normals and second curvatures at the point.*

In accordance with Theorem 3.1, the H -directions of $\{C\}'$ are determined by the tensor $f_{ij} = [f(\sigma)]_{,ij} - [f(\sigma)]_{,i} [f(\sigma)]_{,j}$. From this equation and (4.1),

$$(11.3) \quad f_{;i} = f'(\sigma)\sigma_{;i} + [f''(\sigma) - f'^2(\sigma) + f'(\sigma)]\sigma_{;i}\sigma_{;j}.$$

We first assume that $f'' - f'^2 + f' \neq 0$ and $f' \neq 0$ at a point P . It follows from (11.3) and the remarks below (4.3) that an H -direction of $\{C\}$ at P coincides with an H -direction of $\{C\}'$ if and only if it is a principal direction determined by the tensor $\sigma_{;i}\sigma_{;j}$. But, if $\Delta_1\sigma \neq 0$, these principal directions are $\sigma_{;i}$ and all vectors λ^i such that $\lambda^i\sigma_{;i} = 0$. Of course, the λ^i all lie in the tangent vector spaces of the hypersurfaces $\sigma = \text{const.}$ If one of these vectors λ^i is an H -direction of $\{C\}$ and therefore of $\{C\}'$, it follows from the second italicized statement below Theorem 8.2 that λ^i is tangent to a line of curvature of the hypersurface $\sigma = \text{const.}$ passing through P . If $\sigma_{;i}$ is a common H -direction of $\{C\}$ and $\{C\}'$, it follows from Theorem 9.2 that the remaining common H -directions are the tangents of the lines of curvature of $\sigma = \text{const.}$ at P . If the hypersurfaces $\sigma = \text{const.}$ are parallel, this last situation is realized throughout the space.

If

$$(11.4) \quad f'' - f'^2 + f' = 0$$

and $f' \neq 0$ at P , $\{C\}$ and $\{C\}'$ have the same H -directions at this point. If (11.4) holds throughout V_n , it follows easily that

$$(11.5) \quad e^{f(\sigma)} = \frac{c_1 e^\sigma}{1 - c_2 e^\sigma},$$

where c_1 and c_2 are constants such that the right-hand member of (11.5) is positive. From (2.3), (11.1), and (11.5) we find that

$$(11.6) \quad \frac{a}{ds} = \frac{b}{d\bar{s}} + \frac{c}{ds'}, \quad b, c \neq 0,$$

is equivalent to (11.5).

If $f' = 0$ at P , the H -directions of $\{C\}'$ at this point are either all directions or $\sigma_{;i}$ and all vectors λ^i such that $\lambda^i\sigma_{;i} = 0$ according as $f'' - f'^2 + f'$ does or does not equal zero at P . Some of these results are stated in the next theorem:

THEOREM 11.2. *If the hypersurfaces $\sigma = \text{const.}$ are nonparallel, the similar families of conformal geodesics of V_n which are the images of the geodesics of \bar{V}_n and V'_n will have the same congruences of H -directions if and only if $a/ds = b/d\bar{s} + c/ds'$, where $a, b (\neq 0)$, and $c (\neq 0)$ are constants.*

* If $\Delta_1\sigma = 0$ at P , $\sigma_{;i}$ is tangent to a null vector. In this case it is easy to show that there is a unique null vector ν^i which is not orthogonal to $\sigma_{;i}$ and that any vector which lies in the linear vector space determined by ν^i and all unit vectors normal to $\sigma_{;i}$ is a principal direction determined by $\sigma_{;i}\sigma_{;j}$, and conversely.

As an illustration of the above discussion as well as for its own interest, we consider the following question: What curves in V_n have principal normals equal to the principal normals of their conformal images in \bar{V}_n under the mapping (2.3)?* According to the hypothesis and (2.4),

$$(11.7) \quad \bar{\mu}^i = e^{-\sigma} \mu^i,$$

where μ^i and $\bar{\mu}^i$ are the principal normals of the curve in V_n and \bar{V}_n , respectively. From (2.7) and (11.7), any curve whose principal normal is invariant under (2.3) satisfies the equation†

$$(11.8) \quad \mu^i = \frac{e^{\sigma, m}}{1 - e^{\sigma}} (g^{im} - e^{\xi^i \xi^m}), \quad \sigma \neq 0,$$

where ξ^i is the unit tangent. Hence *the curves whose principal normals are invariant under (2.3) form a family of conformal geodesics similar to the conformal geodesics (2.8) corresponding to the given transformation.* Let (11.8) be the images of the geodesics of V'_n (determined except for a magnification). Then the induced mapping between V_n and V'_n is of the form (11.1) where

$$(11.9) \quad e^{f(\sigma)} = ae^{\sigma}/(e^{\sigma} - 1), \quad \sigma > 0, a > 0; \quad e^{f(\sigma)} = ae^{\sigma}/(1 - e^{\sigma}), \quad \sigma < 0, a > 0.$$

For the region of the coordinate space in which $\sigma > 0$,‡ we investigate the curves whose principal normals in V_n and V'_n are equal under the transformation between these spaces. According to the preceding discussion, these curves are the conformal images of the geodesics of a V''_n . From (11.9), the mapping of V''_n on V_n is $ds'' = e^{F(\sigma)} ds$, where

$$(11.10) \quad e^{F(\sigma)} = \frac{bae^{\sigma}}{|(a - 1)e^{\sigma} + 1|}, \quad b > 0.$$

Of course, $F(\sigma) = f(f(\sigma))$ except for a magnification of V''_n . The conformal correspondence associated with (11.10) will coincide with (2.3) and V''_n with \bar{V}_n if and only if $a = 1$, $b = 1$. In this case, V'_n is uniquely determined by (2.3) and from (11.9) the mapping of V'_n on V_n is given by

$$(11.11) \quad ds' = \frac{e^{\sigma}}{e^{\sigma} - 1} ds, \quad \sigma > 0.$$

Conversely, (11.11) uniquely determines \bar{V}_n and (2.3). This proves the following theorem:

* Curves whose principal normals correspond (but not necessarily with invariant first curvatures) have been considered by V. Modesitt, loc. cit., pp. 326-328.

† At points where $\sigma = 0$, the principal normal is invariant if and only if ξ^i is tangent to $g^{im}\sigma_{,m}$.

‡ In the region where $\sigma < 0$ we simply interchange the roles of V_n and \bar{V}_n .

THEOREM 11.3. *Let (2.3) be a conformal transformation between a region of V_n and a region of \bar{V}_n in which $\sigma > 0$. Then there exists a unique Riemann space V'_n and a unique conformal mapping (11.11) of V'_n on V_n such that the images of the geodesics of V'_n have equal principal normals in V_n and \bar{V}_n and the images of the geodesics of \bar{V}_n have equal principal normals in V_n and V'_n .*

Corresponding to (11.6), we have

$$\frac{1}{ds} = \frac{1}{d\bar{s}} + \frac{1}{ds'}.$$

It follows from this equation or from the above discussion that if V_n , \bar{V}_n , and V'_n are all subjected to the same conformal transformation,

$$ds^* = e^{\sigma} ds, \quad d\bar{s}^* = e^{\sigma} d\bar{s}, \quad ds'^* = e^{\sigma} ds',$$

then the new spaces V_n^* , \bar{V}_n^* , and $V_n'^*$ may replace V_n , \bar{V}_n , and V'_n , respectively, in Theorem 11.3. This means that the triplet of spaces V_n , \bar{V}_n , and V'_n is a conformal triplet with respect to the property stated in Theorem 11.3.

If μ'^i is a principal normal of a curve in V'_n , the transformation corresponding to (2.7) is

$$(11.12) \quad \mu'^i = e^{-2f(\sigma)} [\mu^i - e[f(\sigma)]_{,m} (g^{im} - e\xi^i \xi^m)],$$

where $f(\sigma)$ is defined by (11.9) with $a=1$. Let $\bar{\nu}^i$ and ν'^i represent $\bar{\mu}^i$ and μ'^i considered as vectors in V_n . Then

$$(11.13) \quad \bar{\nu}^i = e^{\sigma} \bar{\mu}^i, \quad \nu'^i = e^{f(\sigma)} \mu'^i.$$

For any curve in V_n and its conformal images in \bar{V}_n and V'_n , we obtain from (2.7), (11.12), and (11.13) that

$$(11.14) \quad \mu^i = \bar{\nu}^i + \nu'^i.$$

In particular, if $\nu'^i = 0$, then $\mu^i = \bar{\nu}^i$; and if $\bar{\nu}^i = 0$, then $\mu^i = \nu'^i$. These properties were used to define V'_n . For the geodesics of V_n , $\mu^i = 0$. It follows from (11.14) that the corresponding images of the geodesics of V_n in \bar{V}_n and V'_n have equal principal normals oppositely directed.

A simple computation shows that

$$e^{\sigma} f_{ij} + e^{f(\sigma)} \sigma_{ij} = 0,$$

so that the H -directions of the similar families of conformal geodesics (2.8) and (11.8) coincide. This also follows from the discussion preceding Theorem 10.2 since (11.9) is of the form (11.5).

12. Conformal transformations with $\sigma_{ij} = \phi g_{ij}$. In previous sections, we have seen that the conformal transformations for which the tensor σ_{ij} with σ

not constant satisfies (8.21) are of a particularly simple and interesting geometric character. We now show that a very large class of V_n 's actually exists which admit such transformations. If we write $\Omega = e^{-\sigma}$, (8.21) becomes

$$(12.1) \quad \Omega_{,ij} = -\phi \Omega g_{ij}.$$

We first investigate the solutions of (12.1) for which

$$(12.2) \quad \Delta_1 \sigma \neq 0 \quad \text{or} \quad \Delta_1 \Omega \neq 0.$$

In this case, the equation

$$(12.3) \quad g^{ij} \Omega_{,i} \theta_{,j} = 0$$

admits $n-1$ independent solutions* ${}_{(p)}\theta$. By means of a suitable coordinate transformation, we obtain

$$(12.4) \quad {}_{(p)}\theta = x^p, \quad \Omega = x^n.$$

It follows from (12.2), (12.3), and (12.4) that

$$(12.5) \quad g^{nn} = \frac{1}{g^{nn}} \neq 0, \quad g^{pn} = 0.$$

In this coordinate system, (12.1) becomes

$$(12.6) \quad \{n \mid ij\} = \phi x^n g_{ij}.$$

We set $i=n, j=p; i=p, j=q; i=n, j=n$ successively in (12.6) and use (12.5). This gives

$$\frac{\partial g_{nn}}{\partial x^p} = 0, \quad -\frac{1}{2} g^{nn} \frac{\partial g_{pq}}{\partial x^n} = \phi x^n g_{pq}, \quad \frac{1}{2} g^{nn} \frac{\partial g_{nn}}{\partial x^n} = \phi x^n g_{nn}.$$

From these equations, we find

$$(12.7) \quad \phi = \phi(x^n),$$

$$(12.8) \quad g^{nn} = - \int 2x^n \phi(x^n) dx^n,$$

$$(12.9) \quad g_{pq} = g^{nn}(x^n) h_{pq}(x^r).$$

The h_{pq} are arbitrary functions of the x^r only. Hence, a V_n admits a solution of (8.21) and (12.2) if and only if the first fundamental form of V_n may be written as

$$(12.10) \quad ds^2 = g_{pq} dx^p dx^q + g_{nn} dx^{n^2},$$

* In this section, the ranges of the indices p, q, r and s, t, u are $1, 2, \dots, n-1$ and $1, 2, \dots, n-2$, respectively.

where the g_{ij} satisfy (12.8) and (12.9). Since $\Delta_1 x^n = g^{nn}(x^n)$, the hypersurfaces $\sigma = \text{const.}$ are parallel. It also follows from (12.8) and (12.9)* (or from the second italicized statement below Theorem 8.2) that these hypersurfaces have indeterminate lines of curvature.

If (12.1) and (12.2) admit other solutions ψ independent of x^n , it follows from (12.7) that $\phi = -a$, where a is a constant, is a necessary condition. We set $i=n, j=p; i=n, j=n; i=p, j=q$ successively in (12.6) and use (12.5), (12.8), and (12.9). As a result, we have

$$(12.11) \quad \frac{\partial^2 \psi}{\partial x^n \partial x^p} - \frac{1}{2} \frac{\partial \psi}{\partial x^p} \frac{d \log g^{nn}}{dx^n} = 0,$$

$$(12.12) \quad \frac{\partial^2 \psi}{\partial x^n^2} - \frac{1}{2} \frac{\partial \psi}{\partial x^n} \frac{d \log g_{nn}}{dx^n} = a \psi g_{nn},$$

$$(12.13) \quad \frac{\partial^2 \psi}{\partial x^p \partial x^q} - \frac{\partial \psi}{\partial x^r} \{r | pq\} - \frac{\partial \psi}{\partial x^n} \{n | pq\} = a \psi g^{nn} h_{pq},$$

where in accordance with (12.8),

$$(12.14) \quad g^{nn} = ax^{n^2} + b.$$

From (12.11), we have

$$(12.15) \quad \psi = (g^{nn})^{1/2} \Lambda(x^p) + \Gamma(x^n).$$

From (12.12) and (12.15), we find

$$(12.16) \quad \frac{d^2 \Gamma}{dx^{n^2}} + \frac{d}{dx^n} (\log (g^{nn})^{1/2}) \frac{d \Gamma}{dx^n} - \frac{a}{g^{nn}} \Gamma = 0.$$

Now, by (12.5) and (12.9),

$$\{r | pq\} = g^{rs} [pq, s] = g_{nn} h^{rs} g^{nn} [pq, s]_h = \{r | pq\}_h$$

where $[pq, s]_h$ and $\{r | pq\}_h$ denote the Christoffel symbols of the first and second kind formed with respect to the form

$$(12.17) \quad ds^2 = h_{pq} dx^p dx^q.$$

Of course, (12.17) is the first fundamental form of each of the hypersurfaces $\sigma = \text{const.}$ except for a magnification. Also, $\{n | pq\} = -axg^{nn} \cdot h_{pq}$. Substituting these results and (12.14) and (12.15) in (12.13) we have

$$(12.18) \quad \Lambda_{;pq} = \left[ab\Lambda + a(g^{nn})^{1/2} \left(\Gamma - x^n \frac{d\Gamma}{dx^n} \right) \right] h_{pq},$$

* Cf. Eisenhart, loc. cit., p. 182.

where the semicolon denotes covariant differentiation with respect to the form (12.17). If $a \neq 0$, it follows that

$$(12.19) \quad (g^{nn})^{1/2} \left(\Gamma - x^n \frac{d\Gamma}{dx^n} \right) = c,$$

where c is a constant. It is easily verified that (12.16) is a consequence of (12.19). If $a=0$, $\Gamma = c_1 x^n + c_2$. In both these cases, Γ satisfies equations similar to (12.1) where the covariant differentiation is with respect to the form (12.17). This shows that *the necessary and sufficient condition that the V_n whose first fundamental form is (12.10) admit more than one independent solution of (8.21) and (12.2) is that ϕ be constant and any hypersurface $\sigma = \text{const.}$ admit a non-constant solution of $\Delta_{pq} = a(b\Delta + c)h_{pq}$.*

We now investigate the solutions of (8.21) for which

$$(12.20) \quad \Delta_1 \sigma = 0 \quad \text{or} \quad \Delta_1 \Omega = 0.$$

We first note that $\phi=0$ is a necessary condition for the existence of such solutions. For, according to (12.1), $(\Delta_1 \Omega)_{,k} = g^{ij}(\Omega_{,ik}\Omega_{,j} + \Omega_{,i}\Omega_{,jk}) = -2\phi\Omega_{,k}$. As a consequence of this equation and (12.20), $\phi=0$. The equation (12.3) admits $n-2$ independent solutions $_{(s)}\theta$ besides the solution Ω . If $_{(n-1)}\theta$ is a solution of

$$(12.21) \quad g^{ij}\Omega_{,i}\theta_{,j} = 1,$$

the θ 's and Ω are a set of n independent variables. By means of the coordinate transformation (12.4) it follows from (12.3) and (12.21) that

$$(12.22) \quad g^{ns} = 0, \quad g^{nn} = 0, \quad g^{(n-1)n} = 1.$$

These results are equivalent to

$$(12.23) \quad g_{s(n-1)} = 0, \quad g_{(n-1)(n-1)} = 0, \quad g_{(n-1)n} = 1.$$

In this coordinate system, (12.1) becomes (12.6) with $\phi=0$. It follows from (12.22) and (12.23) that (12.6) is equivalent to $\partial g_{st}/\partial x^{n-1} = 0$. Hence *a V_n admits a nonconstant solution of (8.21) and (12.20) if and only if the first fundamental form of V_n may be written as*

$$ds^2 = g_{st}(x^u, x^n)dx^s dx^t + 2dx^{n-1}dx^n + g_{nn}dx^{n^2} + g_{sn}dx^s dx^n.$$

In conclusion, we note that if $_{(1)}\Omega, _{(2)}\Omega, \dots, _{(m)}\Omega$ are independent solutions of (12.1), the most general function of the Ω 's which is also a solution of (12.1) is $c_1 _{(1)}\Omega + c_2 _{(2)}\Omega + \dots + c_m _{(m)}\Omega + a$, where the c 's are constants and a is an arbitrary constant or zero according as ϕ is equal to or different from zero.